QUANTITATIVE ISOPERIMETRIC INEQUALITIES IN $\mathbb{H}^n$

VALENTINA FRANCESCHI, GIAN PAOLO LEONARDI, AND ROBERTO MONTI

Abstract. In the Heisenberg group $\mathbb{H}^n$, $n \geq 1$, we prove quantitative isoperimetric inequalities for Pansu's spheres, that are known to be isoperimetric under various assumptions. The inequalities are shown for suitably restricted classes of competing sets and the proof relies on the construction of sub-calibrations.

1. Introduction

Quantitative isoperimetric inequalities in the Euclidean space and in Riemannian manifolds have been an object of intensive studies in recent years. The sharp quantitative isoperimetric inequality in the Euclidean space $\mathbb{R}^n$ states that there exists a constant $C_n > 0$ depending only on the dimension n, such that for any Borel set $F \subset \mathbb{R}^n$ with $\mathscr{L}^n(F) = \mathscr{L}^n(B_1)$, the Lebesgue measure of a unit ball B_1, one has the following estimate for the difference of perimeters

$$P(F) - P(B_1) \geq C_n \inf_{x \in \mathbb{R}^n} \mathscr{L}^n(F \Delta (x + B_1))^2.$$

This inequality is established in its full generality in [13], and proved by different methods in [10, 7]. Several generalization have been recently obtained in Riemannian manifolds (with density), like the Gauss space [2, 6], the n-dimensional sphere [3], and the n-dimensional hyperbolic space [4]. A recurrent technique used in the proofs is based on the regularity theory for perimeter quasiminimizers combined with a penalization trick and a Fuglede-type argument, which essentially exploits the strict positivity of the second variation of the area with respect to non-trivial volume-preserving perturbations (see [7, 1]). With similar arguments, quantitative stability results for global area-minimizing smooth hypersurfaces are obtained in [8], together with more specific results for a subfamily of singular area-minimizing Lawson cones. In this case, due to the presence of a singular point at the vertex of the cone, the proof of the sharp quantitative stability follows a different strategy, that is based on the construction of suitable sub-calibrations (see also [9]).

On the other hand, in the context of subriemannian geometry, and in particular in Carnot groups, very few is known about the optimal constant in the isoperimetric inequality (except for the fact that isoperimetric sets exist and have at least some very weak regularity properties [15]). With the only exception of the Grushin plane [19] (see also [11]), isoperimetric sets have been only partially characterized in the

Date: March 24, 2015.

subriemannian Heisenberg group $\mathbb{H}^n$ (see below) and are not known at all in more general Carnot groups.

The main result of this paper is Theorem 1.1, where we prove the validity of a parameterized family of quantitative isoperimetric inequalities in the subriemannian Heisenberg group $\mathbb{H}^n$.

Before stating the result, we recall some basic definitions. The $2n+1$-dimensional Heisenberg group is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \in \mathbb{N}$, endowed with the group product

$$(z,t) * (\zeta,\tau) = \big(z+\zeta, t+\tau+2\,\mathrm{Im}\langle z,\bar\zeta\rangle\big),$$

where $t,\tau \in \mathbb{R}$, $z,\zeta \in \mathbb{C}^n$ and $\langle z,\bar\zeta\rangle = z_1\bar\zeta_1 + \ldots + z_n\bar\zeta_n$. The bundle of horizontal left-invariant vector fields in $\mathbb{H}^n$ is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}$$

with $z_j = x_j + iy_j$ and $j = 1,\ldots,n$.

The Haar measure of $\mathbb{H}^n$ is the Lebesgue measure $\mathscr{L}^{2n+1}$. The *H-perimeter* of a $\mathscr{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$ in an open set $A \subset \mathbb{H}^n$ is

$$P_H(E,A) = \sup\left\{\int_E \mathrm{div}_H V\,dzdt : V \in C_c^1(A;\mathbb{R}^{2n}), \|V\|_\infty \le 1\right\},$$

where the horizontal divergence of the vector field $V : A \to \mathbb{R}^{2n}$ is

$$\mathrm{div}_H V = \sum_{j=1}^n X_jV_j + Y_jV_{n+j}.$$

We use the notation $\mu_E(A) = P_H(E,A)$ and $P_H(E) = P_H(E,\mathbb{H}^n)$. If $P_H(E) < \infty$ then the open sets mapping $A \mapsto \mu_E(A)$ extends to a Radon measure μ_E on $\mathbb{H}^n$. Moreover, there exists a μ_E-measurable function $\nu_E : \mathbb{H}^n \to \mathbb{R}^{2n}$ such that $|\nu_E| = 1$ μ_E-a.e. and the Gauss-Green integration by parts formula

$$\int_{\mathbb{H}^n} \langle V,\nu_E\rangle\,d\mu_E = -\int_{\mathbb{H}^n} \mathrm{div}_H V\,dzdt \tag{1.1}$$

holds for any $V \in C_c^1(\mathbb{H}^n;\mathbb{R}^{2n})$. Here and hereafter, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^{2n}$.

The isoperimetric problem in the Heisenberg group consists in minimizing H-perimeter of sets with a given fixed volume. By homogeneity with respect to the dilations $(z,t) \mapsto (\lambda z, \lambda^2, t)$ for $\lambda > 0$, this is equivalent to prove existence, uniqueness, and classify the minimizers of the minimum problem

$$\inf\left\{\frac{P_H(E,\mathbb{H}^n)}{\mathscr{L}^{2n+1}(E)^{\frac{2n+1}{2n+2}}} : E \subset \mathbb{H}^n \text{ measurable set with } 0 < \mathscr{L}^{2n+1}(E) < \infty\right\}. \tag{1.2}$$

A set realizing the infimum is called *isoperimetric set*. The existence of isoperimetric sets is established in [15].

In 1983 P. Pansu [21] conjectured that, up to left translation and dilation, the isoperimetric set is

$$E_{\rm isop} = \Big\{(z,t) \in \mathbb{H}^n : |t| < \arccos(|z|) + |z|\sqrt{1-|z|^2},\ |z| < 1\Big\}. \tag{1.3}$$

The conjecture was made for dimension $n = 1$. The boundary of set $E_{\rm isop} \subset \mathbb{H}^1$ can be obtained taking one geodesic for the Carnot-Carathéodory metric joining the south pole $(0, -\pi/2) \in \partial E_{\rm isop}$ to the north pole $(0, \pi/2) \in \partial E_{\rm isop}$ and letting it rotate around the t-axis.

In $\mathbb{H}^1$, Pansu's conjecture is proved assuming either the C^2 regularity of the minimizer [23] or its convexity [20]. In $\mathbb{H}^n$ with $n \geq 1$, the conjecture is proved assuming the axial symmetry of the minimizer [17] or assuming a suitable cylindrical structure [22]. Some observations on the problem can be found in [16] and [14]. See also the book [5] and the lecture notes [18].

By refining the calibration argument of [22] via a sub-calibration, we prove two *quantitative* versions of the Heisenberg isoperimetric inequality for competitors of $E_{\rm isop}$ in half-cylinders.

For any $0 \leq \varepsilon < 1$ we define the half-cylinder

$$C_\varepsilon = \Big\{(z,t) \in \mathbb{H}^n : |z| < 1 \text{ and } t > t_\varepsilon\Big\},$$

where $t_\varepsilon = \varphi(1-\varepsilon)$ with $\varphi(r) = \arccos(r) + r\sqrt{1-r^2}$. The proof provides an inequality with a variable structure, according to whether $\varepsilon = 0$ or $\varepsilon > 0$. A similar construction could be used also in the Euclidean setting for Dido's problem (i.e., for the relative isoperimetric problem in a half-space), and in this case it would provide analogous quantitative estimates for the same classes of competitors. Our main result is the following

Theorem 1.1. Let $F \subset \mathbb{H}^n$, $n \geq 1$, be any measurable set with $\mathscr{L}^{2n+1}(F) = \mathscr{L}^{2n+1}(E_{\rm isop})$.

i) If $F \Delta E_{\rm isop} \subset\subset C_0$ then

$$P_H(F) - P_H(E_{\rm isop}) \geq \frac{n}{240\,\omega_{2n}^2} \mathscr{L}^{2n+1}(F\Delta E_{\rm isop})^3. \tag{1.4}$$

ii) If $F \Delta E_{\rm isop} \subset\subset C_\varepsilon$ for $0 < \varepsilon < 1$, then

$$P_H(F) - P_H(E_{\rm isop}) \geq \frac{n\sqrt{\varepsilon}}{16\,\omega_{2n}} \mathscr{L}^{2n+1}(F\Delta E_{\rm isop})^2. \tag{1.5}$$

Above, ω_{2n} denotes the Lebesgue measure of the Euclidean unit ball in $\mathbb{R}^{2n}$.

In (1.4), the asymmetry index $\mathscr{L}^{2n+1}(F\Delta E_{\rm isop})$ appears with the power 3. In (1.5), the power is 2 but there is a constant that vanishes with ε. The quantitative isoperimetric inequality in $\mathbb{R}^n$ [13] shows that the optimal power is 2.

The sub-calibration is constructed in the following way. The set $E_{\rm isop} \cap C_\varepsilon$ can be foliated by a family of hypersurfaces with constant H-mean curvature that decreases from 1, the H-curvature of $\partial E_{\rm isop}$, to 0, the curvature of the surface $\{t = t_\varepsilon\}$. The

velocity of the decrease depends on the parameter ε. The horizontal unit normal to the leaves gives the sub-calibration.

The H-mean curvature is defined in the following way. Let $\Sigma \subset \mathbb{H}^n$ be a hypersurface that is locally given by the zero set of a function $u \in C^1$ such that $|\nabla_H u| \neq 0$ on Σ, where

$$\nabla_H u = \big(X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u\big) \tag{1.6}$$

is the horizontal gradient of u. Then we define the H-mean curvature of Σ at the point $(z,t) \in \Sigma$ as

$$H_\Sigma(z,t) = \frac{1}{2n} \mathrm{div}_H \Big(\frac{\nabla_H u(z,t)}{|\nabla_H u(z,t)|} \Big). \tag{1.7}$$

The definition depends on a choice of sign. We shall work with orientable embedded hypersurfaces and so we can choose the positive sign, $H(z,t) \geq 0$. Then, the boundary of E_{isop} has constant H-mean curvature 1. For a set $E = \{(z,t) \in \mathbb{H}^n : u(z,t) > 0\}$ the horizontal normal ν_E in the Gauss-Green formula (1.1) is given on ∂E by the vector

$$\nu_E = \frac{\nabla_H u}{|\nabla_H u|}.$$

The proof of Theorem 1.1 relies on the construction described in the following result.

Theorem 1.2. Let $0 \leq \varepsilon < 1$. There exists a continuous function $u : C_\varepsilon \to \mathbb{R}$ with level sets $\Sigma_s = \big\{(z,t) \in C_\varepsilon : u(z,t) = s\big\}$, $s \in \mathbb{R}$, such that:

i) $u \in C^1(C_\varepsilon \cap E_{\mathrm{isop}}) \cap C^1(C_\varepsilon \setminus E_{\mathrm{isop}})$ and $\nabla_H u / |\nabla_H u|$ is continuously defined on $C_\varepsilon \setminus \{z = 0\}$;

ii) $\bigcup_{s>1} \Sigma_s = C_\varepsilon \cap E_{\mathrm{isop}}$ and $\bigcup_{s \leq 1} \Sigma_s = C_\varepsilon \setminus E_{\mathrm{isop}}$;

iii) Σ_s is a hypersurface of class C^2 with constant H-mean curvature $H_{\Sigma_s} = 1/s$ for $s > 1$ and $H_{\Sigma_s} = 1$ for $s \leq 1$;

iv) For any point $(z, \varphi(|z|) - t) \in \Sigma_s$ with $s > 1$ we have

$$1 - H_{\Sigma_s}(z, \varphi(|z|) - t) \geq \frac{1}{20} t^2 \quad \text{when } \varepsilon = 0. \tag{1.8}$$

and

$$1 - H_{\Sigma_s}(z, \varphi(|z|) - t) \geq \frac{\sqrt{\varepsilon}}{4} t \quad \text{when } 0 < \varepsilon < 1, \tag{1.9}$$

The estimates (1.8) and (1.9) are the basis of the two inequalities (1.4) and (1.5), respectively.

2. Proof of Theorem 1.2

In $C_\varepsilon \setminus E_{\mathrm{isop}}$, the leaves Σ_s are vertical translations of the top part of the boundary $\partial E_{\mathrm{isop}}$. In $C_\varepsilon \cap E_{\mathrm{isop}}$, the leaves Σ_s are constructed in the following way: the surface $\partial E_{\mathrm{isop}}$ is first dilated by a factor larger than 1, and then it is translated downwards in such a way that, after the two operations, the sphere $\{(z,t) \in \partial E_{\mathrm{isop}} : t = t_\varepsilon\}$ with $t_\varepsilon = \varphi(1 - \varepsilon)$ remains fixed.

The profile function of the set $E_{\rm isop}$ is the function $\varphi : [0,1] \to \mathbb{R}$

$$\varphi(r) = \arccos(r) + r\sqrt{1-r^2} \qquad 0 \le r \le 1. \tag{2.1}$$

Its first and second order derivatives are

$$\varphi'(r) = \frac{-2r^2}{\sqrt{1-r^2}} \quad \text{and} \quad \varphi''(r) = \frac{2r(r^2-2)}{(1-r^2)^{3/2}}, \quad 0 \le r < 1. \tag{2.2}$$

Notice that $\varphi'''(0) = -4$. We also need the function $\psi : [0,1) \to \mathbb{R}$

$$\psi(r) = 2\varphi(r) - r\varphi'(r) = 2\left(\frac{r}{\sqrt{1-r^2}} + \arccos(r)\right). \tag{2.3}$$

Its derivative is

$$\psi'(r) = \varphi'(r) - r\varphi''(r) = \frac{2r^2}{(1-r^2)^{3/2}}. \tag{2.4}$$

We start the construction of the function u. On the set $C_\varepsilon \setminus E_{\rm isop}$ we let

$$u(z,t) = \varphi(|z|) - t + 1, \quad (z,t) \in C_\varepsilon \setminus E_{\rm isop}. \tag{2.5}$$

Notice that $u(z, \varphi(|z|)) = 1$ for all $|z| < 1$. We define the function u in the set

$$D_\varepsilon = C_\varepsilon \cap E_{\rm isop} = \big\{(z,t) \in E_{\rm isop} : |z| < 1-\varepsilon,\ t_\varepsilon < t < \varphi(|z|)\big\}.$$

We use the short notation $r = |z|$ and $r_\varepsilon = 1 - \varepsilon$. Let $F_\varepsilon : D_\varepsilon \times (1,\infty) \to \mathbb{R}$ be the function

$$F_\varepsilon(z,t,s) = s^2\big(\varphi(r/s) - \varphi(r_\varepsilon/s)\big) + t_\varepsilon - t.$$

We claim that for any point $(z,t) \in D_\varepsilon$ there exists a unique $s > 1$ such that $F_\varepsilon(z,t,s) = 0$. In this case, we can define the function $u(z,t) : D_\varepsilon \to \mathbb{R}$ letting

$$F_\varepsilon(z,t,s) = 0 \quad \text{if and only if} \quad s = u(z,t). \tag{2.6}$$

We prove the claim. For any $(z,t) \in D_\varepsilon$ we have

$$\lim_{s\to 1^+} F_\varepsilon(z,t,s) = \varphi(r) - t > 0.$$

Moreover, with a second order Taylor expansion of φ based on (2.2) we see that

$$\lim_{s\to\infty} F_\varepsilon(z,t,s) = t_\varepsilon - t < 0.$$

Since $s \mapsto F_\varepsilon(z,t,s)$ is continuous, this proves the existence of a solution of $F_\varepsilon(z,t,s) = 0$. By (2.3), the derivative in s of F_ε is

$$\partial_s F_\varepsilon(z,t,s) = s\big(\psi(r/s) - \psi(r_\varepsilon/s)\big), \tag{2.7}$$

and thus by (2.4) we deduce that $\partial_s F_\varepsilon(z,t,s) < 0$. This proves the uniqueness.

We prove claim iii). Namely, we prove that for any point $(z,t) \in \Sigma_s$ with $s > 1$ and $z \neq 0$, the H-mean curvature of Σ_s at (z,t) is

$$H_{\Sigma_s}(z,t) = -\frac{1}{2n}\mathrm{div}_H\Big(\frac{\nabla_H u}{|\nabla_H u|}\Big) = \frac{1}{s}. \tag{2.8}$$

We are using definition (1.6) with a minus sign in order to have a positive curvature. The claim when $s \le 1$ is analogous because Σ_s is a vertical translation of the top part of $\partial E_{\rm isop}$.

By the implicit function theorem, the derivatives of u can be computed from the partial derivatives of F_ε. Using $\partial_{x_i} r = x_i/r$ and $\partial_{y_i} r = y_i/r$, with $i = 1, \dots, n$ and $z = (x_1 + iy_1, \dots, x_n + iy_n)$, we find

$$\partial_{x_i} F_\varepsilon(z,t,s) = \frac{sx_i}{r}\varphi'(r/s) \quad \text{and} \quad \partial_{y_i} F_\varepsilon(z,t,s) = \frac{sy_i}{r}\varphi'(r/s). \tag{2.9}$$

Letting $s = u(z,t)$, thanks to (2.6), (2.7), (2.9), and (2.2) we obtain

$$\partial_{x_i} u(z,t) = -\frac{\partial_{x_i} F_\varepsilon(z,t,s)}{\partial_s F_\varepsilon(z,t,s)} = \frac{2rx_i}{s\sqrt{s^2-r^2}\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)}, \tag{2.10}$$

$$\partial_{y_i} u(z,t) = -\frac{\partial_{x_i} F_\varepsilon(z,t,s)}{\partial_s F_\varepsilon(z,t,s)} = \frac{2ry_i}{s\sqrt{s^2-r^2}\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)}, \tag{2.11}$$

$$\partial_t u(z,t) = -\frac{\partial_t F_\varepsilon(z,t,s)}{\partial_s F_\varepsilon(z,t,s)} = \frac{1}{s\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)}, \tag{2.12}$$

and thus

$$\partial_{x_i} u = 2x_i \frac{r}{\sqrt{s^2-r^2}}\partial_t u \quad \text{and} \quad \partial_{y_i} u = 2y_i \frac{r}{\sqrt{s^2-r^2}}\partial_t u. \tag{2.13}$$

It is then immediate to compute

$$X_i u = \partial_{x_i} u + 2y_i \partial_t u = \frac{2rx_i + 2y_i\sqrt{s^2-r^2}}{s\sqrt{s^2-r^2}\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)},$$

$$Y_i u = \partial_{y_i} u - 2x_i \partial_t u = \frac{2ry_i - 2x_i\sqrt{s^2-r^2}}{s\sqrt{s^2-r^2}\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)},$$

and the squared length of the horizontal gradient of u in D_ε is

$$\begin{aligned} |\nabla_H u|^2 &= \sum_{i=1}^n (X_i u)^2 + (Y_i u)^2 \\ &= \sum_{i=1}^n \frac{4r^2(x_i^2+y_i^2) + 4(x_i^2+y_i^2)(s^2-r^2)}{s^2(s^2-r^2)\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)^2} \\ &= \frac{4r^2}{(s^2-r^2)\big(\psi(r/s) - \psi(r_\varepsilon/s)\big)^2}. \end{aligned}$$

Note that $|\nabla_H u(z,t)| = 0$ if and only if $z = 0$. So for any $(z,t) \in D_\varepsilon$ with $z \neq 0$ we have

$$a_i(z,t) = -\frac{X_i u}{|\nabla_H u|} = \frac{rx_i + y_i\sqrt{s^2-r^2}}{rs} = \frac{x_i}{s} + y_i\frac{\sqrt{s^2-r^2}}{rs} \tag{2.14}$$

and

$$b_i(z,t) = -\frac{Y_i u}{|\nabla_H u|} = \frac{ry_i - x_i\sqrt{s^2-r^2}}{rs} = \frac{y_i}{s} - x_i\frac{\sqrt{s^2-r^2}}{rs}. \tag{2.15}$$

If $(z,t) \in E_{\text{isop}}$ tends to $(\bar z, \bar t) \in \partial E_{\text{isop}}$ with $\bar t > 0$ and $\bar z \neq 0$, then $s = u(z,t)$ converges to 1, and from (2.14) and (2.15) we see that

$$\lim_{(z,t)\to(\bar z,\bar t)} \frac{\nabla_H u(z,t)}{|\nabla_H u(z,t)|} = -\Big(\bar x + \bar y\frac{\sqrt{1-|\bar z|^2}}{|\bar z|}, \bar y - \bar x\frac{\sqrt{1-|\bar z|^2}}{|\bar z|}\Big) = \frac{\nabla_H u(\bar z,\bar t)}{|\nabla_H u(\bar z,\bar t)|},$$

where the right hand side is computed using the definition (2.5) of u. This ends the proof of claim i).

Claim ii) is clear. We prove claim iii). The auxiliary function $w(r,s) = \sqrt{s^2-r^2}/rs$ satisfies

$$\partial_{x_i} w = \frac{x_i}{r}\partial_r w + \partial_{x_i} u\, \partial_s w, \quad \partial_{y_i} w = \frac{y_i}{r}\partial_r w + \partial_{y_i} u\, \partial_s w, \quad \partial_s w = \frac{r}{s^2\sqrt{s^2-r^2}}. \tag{2.16}$$

By (2.14), (2.15), (2.13), and (2.16) we obtain

$$\begin{aligned}
X_i a_i + Y_i b_i &= \partial_{x_i} a_i + 2y_i \partial_t a_i + \partial_{y_i} b_i - 2x_i \partial_t b_i \\
&= \frac{1}{s} - \frac{x_i}{s^2}\partial_{x_i} u + y_i\Big(\frac{x_i}{r}\partial_r w + \partial_{x_i} u\,\partial_s w\Big) + 2y_i\Big(-\frac{x_i}{s^2}\partial_t u + y_i \partial_s w\,\partial_t u\Big) \\
&\quad + \frac{1}{s} - \frac{y_i}{s^2}\partial_{y_i} u - x_i\Big(\frac{y_i}{r}\partial_r w + \partial_{y_i} u\,\partial_s w\Big) - 2x_i\Big(-\frac{y_i}{s^2}\partial_t u - x_i\partial_s w\,\partial_t u\Big) \\
&= \frac{2}{s} - \frac{x_i\partial_{x_i} u + y_i \partial_{y_i} u}{s^2} + 2(x_i^2+y_i^2)\partial_s w\,\partial_t u \\
&= \frac{2}{s} - \frac{x_i\partial_{x_i} u + y_i \partial_{y_i} u}{s^2} + \frac{2r(x_i^2+y_i^2)\partial_t u}{s^2\sqrt{s^2-r^2}} = \frac{2}{s}.
\end{aligned}$$

Summing over $i = 1,\dots,n$ and dividing by $2n$, we obtain (2.8).

We prove claim iv). We fix a point z with $|z| < 1-\varepsilon$ and for $0 \le t < \varphi(|z|) - t_\varepsilon$ we define the function

$$f_z(t) = u(z, \varphi(|z|) - t) = s = \frac{1}{H_{\Sigma_s}}, \tag{2.17}$$

where $s \ge 1$ is uniquely determined by $(z, \varphi(|z|) - t) \in \Sigma_s$. The function $t \mapsto f_z(t)$ is increasing and $f_z(0) = 1$

By (2.12), the function f_z solves the differential equation

$$f_z'(t) = -\partial_t u(z, \varphi(|z|) - t) = \frac{1}{f_z(t)\big(\psi(r_\varepsilon/f_z(t)) - \psi(r/f_z(t))\big)}$$

for all $0 < t < \varphi(|z|) - t_\varepsilon$, and since, by (2.4), ψ is strictly increasing, f_z solves the differential inequality

$$f_z'(t) \ge \frac{1}{f_z(t)\big(\psi(r_\varepsilon/f_z(t)) - \pi\big)}.$$

On the other hand, for any $s>1$ we have

$$
\begin{aligned}
s\big(\psi(r_\varepsilon/s)-\pi\big) &= s\int_0^{r_\varepsilon/s} \psi'(r)\,dr \\
&= s\int_0^{r_\varepsilon/s} \frac{2r^2}{(1-r^2)^{3/2}}\,dr \\
&\le r_\varepsilon \int_0^{r_\varepsilon/s} \frac{2r}{(1-r^2)^{3/2}}\,dr \\
&= 2r_\varepsilon\Big((1-(r_\varepsilon/s)^2)^{-1/2}-1\Big) \\
&\le \frac{2}{\sqrt{s-r_\varepsilon}}.
\end{aligned}
\tag{2.18}
$$

In the case $\varepsilon=0$ we have $r_\varepsilon=1$ and inequality (2.18) reads

$$
s\big(\psi(1/s)-\pi\big) \le \frac{2}{\sqrt{s-1}}.
$$

Hence, the function f_z satisfies the differential inequality

$$
f_z'(t) \ge \frac{1}{2}\sqrt{f_z(t)-1}, \qquad t>0.
$$

An integration with $f_z(0)=1$ gives $f_z(t)\ge 1+t^2/16$, and thus by the relation (2.17) and by the bound $t<\pi/2$ we find

$$
1-H_{\Sigma_s}(z,\varphi(|z|)-t) = 1-\frac{1}{f_z(t)} \ge \frac{t^2}{16+t^2} \ge \frac{1}{20}t^2.
$$

This is claim (1.8).

When $0<\varepsilon<1$, inequality (2.18) implies

$$
s\big(\psi(r_\varepsilon/s)-\pi\big) \le \frac{2}{\sqrt{\varepsilon}},
$$

and thus $f_z'(t)\ge\sqrt{\varepsilon}/2$, that gives $f_z(t)\ge 1+t\sqrt{\varepsilon}/2$. In this case, we find

$$
1-H_{\Sigma_s}(z,\varphi(|z|)-t) = 1-\frac{1}{f_z(t)} \ge \frac{2\sqrt{\varepsilon}t}{4+\pi} \ge \frac{\sqrt{\varepsilon}}{4}t.
$$

This is claim (1.9). This finishes the proof of Theorem 1.2.

3. Proof of Theorem 1.1

In this section, we prove the quantitative isoperimetric estimates (1.4) and (1.5).

Let $u: C_\varepsilon \to \mathbb{R}$, $0\le\varepsilon<1$, be the function given by Theorem 1.2 and let $\Sigma_s=\{(z,t)\in C_\varepsilon : u(z,t)=s\}$ be the leaves of the foliation, $s\in\mathbb{R}$. On $C_\varepsilon\setminus\{|z|=0\}$ we define the vector field $X: C_\varepsilon\setminus\{|z|=0\}\to\mathbb{R}^{2n}$ by

$$
X = -\frac{\nabla_H u}{|\nabla_H u|}.
$$

Both u and X depend on ε. In particular, X satisfies the following properties:

i) $|X| = 1$;
ii) for $(z,t) \in \partial E_{\rm isop} \cap C_\varepsilon$ we have $X(z,t) = -\nu_{E_{\rm isop}}(z,t)$, the horizontal unit normal to $\partial E_{\rm isop}$.
iii) For any point $(z,t) \in \Sigma_s$, $s \in \mathbb{R}$, we have,

$$\frac{1}{2n}\mathrm{div}_H X(z,t) = H_{\Sigma_s}(z,t) \le H_{\Sigma_0} = 1. \tag{3.1}$$

We start the proof. Let $F \subset \mathbb{H}^n$ be a set with finite H-perimeter such that $\mathscr{L}^{2n+1}(F) = \mathscr{L}^{2n+1}(E_{\rm isop})$ and $F\Delta E_{\rm isop} \subset\subset C_\varepsilon$. By Theorem 2.5 in [12], we can without loss of generality assume that ∂F is of class C^∞. For $\delta > 0$, let $E^\delta_{\rm isop} = \{(z,t) \in E_{\rm isop} \ : \ |z| > \delta\}$. By (3.1) and by the Gauss-Green formula (1.1), we have

$$\begin{aligned}\mathscr{L}^{2n+1}(E^\delta_{\rm isop} \setminus F) &= \int_{E^\delta_{\rm isop}\setminus F} 1\, dzdt \ge \int_{E^\delta_{\rm isop}\setminus F} \frac{\mathrm{div}_H X}{2n}\, dzdt \\ &= \frac{1}{2n}\left\{\int_{\partial F\cap E^\delta_{\rm isop}} \langle X, \nu_F\rangle d\mu_F - \int_{(\partial E^\delta_{\rm isop})\setminus F} \langle X, \nu_{E^\delta_{\rm isop}}\rangle d\mu_{E^\delta_{\rm isop}}\right\}.\end{aligned}$$

Observe that $\mu_{E^\delta_{\rm isop}} = \mu_{E_{\rm isop}} \llcorner \{|z| > \delta\} + \mu_{\{|z|>\delta\}} \llcorner E_{\rm isop}$ and $\mu_{\{|z|>\delta\}}(E_{\rm isop}) \le C\delta^{2n-1}$. Letting $\delta \to 0^+$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathscr{L}^{2n+1}(E_{\rm isop} \setminus F) &\ge \frac{1}{2n}\left\{\int_{\partial F\cap E_{\rm isop}} \langle X, \nu_F\rangle d\mu_F - \int_{(\partial E_{\rm isop})\setminus F} \langle X, \nu_{E_{\rm isop}}\rangle d\mu_{E_{\rm isop}}\right\} \\ &\ge \frac{1}{2n}\left\{\mu_{E_{\rm isop}}(C_\varepsilon \setminus F) - \mu_F(E_{\rm isop})\right\} \\ &= \frac{1}{2n}\{P_H(E_{\rm isop}, C_\varepsilon \setminus F) - P_H(F, E_{\rm isop})\}.\end{aligned} \tag{3.2}$$

By a similar computation we also have

$$\mathscr{L}^{2n+1}(F \setminus E_{\rm isop}) = \int_{F\setminus E_{\rm isop}} 1\, dzdt = \int_{F\setminus E_{\rm isop}} \frac{\mathrm{div}_H X}{2n}\, dzdt \tag{3.3}$$

$$\begin{aligned}&= \frac{1}{2n}\left\{\int_{\partial E_{\rm isop}\cap F} \langle X, \nu_{E_{\rm isop}}\rangle d\mu_{E_{\rm isop}} - \int_{(\partial F)\setminus E_{\rm isop}} \langle X, \nu_F\rangle d\mu_F\right\} \\ &\le \frac{1}{2n}\left\{\mu_F(C_\varepsilon \setminus E_{\rm isop}) - \mu_{E_{\rm isop}}(F)\right\} \\ &= \frac{1}{2n}\{P_H(F, C_\varepsilon \setminus E_{\rm isop}) - P_H(E_{\rm isop}, F)\}.\end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned}\int_{E_{\rm isop}\setminus F} \frac{\mathrm{div}_H X}{2n}\, dzdt &= \int_{E_{\rm isop}\setminus F} \Big(1 + \Big(\frac{\mathrm{div}_H X}{2n} - 1\Big)\Big)\, dzdt \\ &= \mathscr{L}^{2n+1}(E \setminus F) - \int_{E_{\rm isop}\setminus F} \Big(1 - \frac{\mathrm{div}_H X}{2n}\Big)\, dzdt \\ &= \mathscr{L}^{2n+1}(E_{\rm isop} \setminus F) - \mathscr{G}(E_{\rm isop} \setminus F),\end{aligned}$$

where

$$\mathscr{G}(E_{\rm isop} \setminus F) = \int_{E_{\rm isop}\setminus F} \Big(1 - \frac{{\rm div}_H X}{2n}\Big)\, dzdt.$$

From (3.2) and (3.3), we obtain

$$\begin{aligned}\frac{1}{2n}\left\{P_H(E_{\rm isop}, C_\varepsilon \setminus F) - P_H(F, E_{\rm isop})\right\} &\le \int_{E_{\rm isop}\setminus F} \frac{{\rm div}_H X}{2n}\, dzdt \\ &= \mathscr{L}^{2n+1}(E_{\rm isop}\setminus F) - \mathscr{G}(E_{\rm isop}\setminus F) \\ &= \mathscr{L}^{2n+1}(F\setminus E_{\rm isop}) - \mathscr{G}(E_{\rm isop}\setminus F) \\ &\le \frac{1}{2n}\left\{P_H(F, C_\varepsilon\setminus E_{\rm isop}) - P_H(E_{\rm isop}, F)\right\} - \mathscr{G}(E_{\rm isop}\setminus F),\end{aligned}$$

that is equivalent to

$$P_H(F) - P_H(E_{\rm isop}) \ge 2n\mathscr{G}(E_{\rm isop}\setminus F). \tag{3.5}$$

For any z with $|z| < 1-\varepsilon$, we define the vertical sections $E^z_{\rm isop} = \{t \in \mathbb{R} : (z,t) \in E_{\rm isop}\}$ and $F^z = \{t \in \mathbb{R} : (z,t) \in F\}$. By Fubini-Tonelli theorem, we have

$$\begin{aligned}\mathscr{G}(E_{\rm isop}\setminus F) &= \int_{E_{\rm isop}\setminus F}\Big(1 - \frac{{\rm div}_H X}{2n}\Big)\, dzdt \\ &= \int_{\{|z|<1\}}\int_{E^z_{\rm isop}\setminus F^z}\Big(1 - \frac{{\rm div}_H X(z,t)}{2n}\Big)\, dt\, dz.\end{aligned}$$

The function $t \mapsto {\rm div}_H X(z,t)$ is increasing, and thus letting $m(z) = \mathscr{L}^1(E^z_{\rm isop}\setminus F^z)$, by monotonicity we obtain

$$\begin{aligned}\mathscr{G}(E_{\rm isop}\setminus F) &\ge \int_{\{|z|<1\}}\int_{\varphi(|z|)-m(z)}^{\varphi(|z|)}\Big(1 - \frac{{\rm div}_H X(z,t)}{2n}\Big)\, dt\, dz \\ &= \int_{\{|z|<1\}}\int_0^{m(z)}\Big(1 - \frac{1}{f_z(t)}\Big)\, dt\, dz,\end{aligned}$$

where $f_z(t) = u(z, \varphi(|z|) - t)$ is the function introduced in (2.17).

By (1.8), when $\varepsilon = 0$ the function f_z satisfies the estimate $1 - 1/f_z(t) \ge t^2/20$, and by Hölder inequality we find

$$\begin{aligned}\mathscr{G}(E_{\rm isop}\setminus F) &\ge \frac{1}{20}\int_{\{|z|<1\}}\int_0^{m(z)} t^2 dt\, dz \\ &= \frac{1}{60}\int_{\{|z|<1\}} m(z)^3\, dz \\ &\ge \frac{1}{60\omega_{2n}^2}\Big(\int_{\{|z|<1\}} m(z)\, dz\Big)^3 \\ &= \frac{1}{480\omega_{2n}^2}\mathscr{L}^{2n+1}(E_{\rm isop}\Delta F)^3.\end{aligned} \tag{3.6}$$

From (3.6) and (3.5) we obtain (1.4).

By (1.9), when $0<\varepsilon<1$ the function f_z satisfies the estimate $1-1/f_z(t)\geq\sqrt{\varepsilon}t/4$ and we find

$$\begin{aligned}\mathscr{G}(E_{\mathrm{isop}}\setminus F) &\geq \frac{\sqrt{\varepsilon}}{4}\int_{\{|z|<1\}}\int_0^{m(z)} t\,dt\,dz\\ &= \frac{\sqrt{\varepsilon}}{8}\int_{\{|z|<1\}} m(z)^2\,dz\\ &\geq \frac{\sqrt{\varepsilon}}{8\omega_{2n}}\Big(\int_{\{|z|<1\}} m(z)\,dz\Big)^2\\ &= \frac{\sqrt{\varepsilon}}{32\omega_{2n}}\mathscr{L}^{2n+1}(E_{\mathrm{isop}}\Delta F)^2.\end{aligned}\tag{3.7}$$

From (3.7) and (3.5) we obtain claim (1.5).

References

[1] E. Acerbi, N. Fusco, M. Morini, Minimality via second variation for a nonlocal isoperimetric problem. Comm. Math. Phys., (2013) 322(2), 515–557.

[2] M. Barchiesi, A. Brancolini, V. Julin, Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality. Preprint (2014).

[3] V. Bögelein, F. Duzaar, N. Fusco, A quantitative isoperimetric inequality on the sphere. Preprint (2013).

[4] V. Bögelein, F. Duzaar, C. Scheven, A sharp quantitative isoperimetric inequality in hyperbolic n-space. Preprint (2014).

[5] L. Capogna, D. Danielli, S. D. Pauls, J. Tyson, An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. Progress in Mathematics, 259. Birkhäuser Verlag, Basel, 2007. xvi+223 pp.

[6] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, On the isoperimetric deficit in Gauss space. Amer. J. Math., (2011) 133(1), 131–186.

[7] M. Cicalese, G. P. Leonardi, A selection principle for the sharp quantitative isoperimetric inequality. Arch. Rat. Mech. Anal. 206 (4) (2012), 617–643.

[8] G. De Philippis, F. Maggi, Sharp stability inequalities for the Plateau problem. J. Diff. Geom. (2014) 96(3), 399–456.

[9] G. De Philippis, E. Paolini, A short proof of the minimality of Simons cone. Rend. Sem. Mat. Univ. Padova (2009), 121, 233–241.

[10] A. Figalli, F. Maggi, A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities. Inv. Math., (2010) 182(1), 167–211.

[11] V. Franceschi, R. Monti, Isoperimetric problem in H-type groups and Grushin spaces. Preprint (2014).

[12] B. Franchi, R. Serapioni, F. Serra Cassano, Meyers-Serrin Type Theorems and Relaxation of Variational Integrals Depending Vector Fields. Houston Journal of Mathematics 22 (1996), 859–889.

[13] N. Fusco, F. Maggi, A. Pratelli, The sharp quantitative isoperimetric inequality, Ann. of Math., 168 (2008), no. 3, 941–980.

[14] G. P. Leonardi, S. Masnou, On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$. Ann. Mat. Pura Appl. (4) 184 (2005), 533–553.

[15] G. P. Leonardi, S. Rigot, Isoperimetric sets on Carnot groups. Houston J. Math. 29 (2003), 609–637.

[16] R. Monti, Brunn–Minkowski and isoperimetric inequality in the Heisenberg group. Ann. Acad. Sci. Fenn. Math. 28 (2003), 99–109.

[17] R. Monti, Heisenberg isoperimetric problem. The axial case. Adv. Calc. Var. 1 (2008), no. 1, 93–121.

[18] R. Monti, Isoperimetric problem and minimal surfaces in the Heisenberg group, Lecture notes of the ERC School *Geometric Measure Theory and Real Analysis*, 57–130, Edizioni SNS Pisa, 2014–2015.

[19] R. Monti, D. Morbidelli, Isoperimetric inequality in the Grushin plane. J. Geom. Anal., (2004), 14(2), 355–368.

[20] R. Monti, M. Rickly, Convex isoperimetric sets in the Heisenberg group. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), no. 2, 391–415.

[21] P. Pansu, An isoperimetric inequality on the Heisenberg group. Conference on differential geometry on homogeneous spaces (Turin, 1983). Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1983), 159–174.

[22] M. Ritoré, A proof by calibration of an isoperimetric inequality in the Heisenberg group $\mathbb{H}^n$. Calc. Var. Partial Differential Equations 44 (2012), no. 1-2, 47–60.

[23] M. Ritoré, C. Rosales, Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$. Adv. Math. 219 (2008), no. 2, 633–671.